

STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM
AFDELING ZUIVERE WISKUNDE

ZW 1969 - 012

Nowhere differentiable continuous functions

with an extended list of references

by

P. van Emde Boas



October 1969

ZW

BIBLIOTHEEK MATHEMATISCH CENTRUM
 AMSTERDAM

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam,
The Netherlands.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications; it is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O) and the Central Organization for Applied Scientific Research in the Netherlands (T.N.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

§0. Introduction

Nowhere differentiable continuous functions have been discovered, independently, by WEIERSTRASS [8, 96], CELERIER [15], and BOLZANO [42, 69]. This discovery in the nineteenth century created a new subject in mathematics, which occupies mathematicians up to now.

In the Mathematical Reviews of the last years one easily finds in each volume a couple of references to papers treating this subject.

In general these papers contain a new elementary example of a non-differentiable function or a more elegant proof that an earlier constructed function is non-differentiable.

I became interested in this problem while trying to find an elementary proof for the non-differentiability of the function $f(x) = \sum_{n=0}^{\infty} 2^{-n} \sin(2^n \pi x)$.

This function is of the same type as CELERIER's function $f(x) = \sum_{n=0}^{\infty} a^{-n} \sin(a^n \pi x)$. In his original example CELERIER takes for a an even integer > 1000 and his proof fails for $a = 2$.

The non differentiability of this function can be deducted from a theorem of HARDY [36] but this proof cannot be considered to be elementary.

After finding the proof which is described in §1, I looked for similar methods in the literature. None of the papers I was able to study gives the non-differentiability of this special function $f(x)$ as an easy corollary with exception of HARDY's above mentioned paper. However, none of the principles on which the argument is based, can be considered to be new or unknown. The result thus demonstrates that by simply considering "more" terms in a development classical methods can lead to better results.

In §2 I present a list of the references I was able to study. I also mention references I found in other publications, the original papers of which were not available in Amsterdam. This list of references is preceded by a survey of the history of the problem and a description of the methods used in solving it. Information of this kind is rarely given but can be found in the papers 36, 40, 46 and 56.

§1. An elementary proof for the non-differentiability of

$$\underline{f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \sin(2^n \pi x)}$$

Lemma 1: Let $f(x)$ be a real-valued function on the real line $\mathbb{R}$ and let $x_0 \in \mathbb{R}$. Then f has a finite derivative $f'(x_0)$ in $x_0 \in \mathbb{R}$ iff the following condition holds:

for any $\varepsilon > 0$ there exist a $\delta > 0$ such that

$$x_0 - \delta \leq a_1, a_2 \leq x_0 \leq b_1, b_2 \leq x_0 + \delta, a_1 \neq b_1, a_2 \neq b_2$$

implies

$$\left| \frac{f(b_1) - f(a_1)}{b_1 - a_1} - \frac{f(b_2) - f(a_2)}{b_2 - a_2} \right| < \varepsilon$$

Proof: Suppose first that the condition is fulfilled. It then follows that the difference-quotient

$$\frac{f(x) - f(x_0)}{x - x_0}$$

satisfies a Cauchy condition for $x \rightarrow x_0$, hence it has a finite limit for $x \rightarrow x_0$, and f is differentiable in x_0 with a finite derivative.

Reversily if f has a finite derivative $f'(x_0)$ we have

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x - x_0) \quad \text{for } x \rightarrow x_0$$

This implies:

$$\lim_{a, b \rightarrow x_0} \frac{f(b) - f(a)}{b - a} = f'(x_0)$$

provided that x_0 is in between a and b ! The proof of this fact is completely elementary

The condition given by Lemma 1 is used by many authors in this field of mathematics. It is generally supposed to be not commonly

known. It appears however already in 1882 in the works of T.J. STIELTJES [87].

Lemma 2: Let $g(x)$ be the function

$$g(x) = \sin 2\pi x + \sin \pi x + (2-\sqrt{2}) \sin(\pi x/2)$$

and let x_0 be some real number.

Then there exists a real number y such that

$$y \leq x \leq y + 1 \quad \text{and} \quad |g(y)| > \frac{\pi^2}{16} + 0,05$$

Proof: The function $g(x)$ is tabulated in Table 1 for $x = 0 (1/8)$ up to 4. $g(x)$ is periodic modulo 4 and it is therefore sufficient to find points $y_0, \dots, y_N \in [0, 4]$ so that

$$y_i - y_{i-1} < 1 \quad \text{for } i = 1, \dots, N$$

and

$$y_0 + 4 - y_N < 1$$

where we have

$$|g(y_i)| > 0,7 > \pi^2/16 + 0,05$$

We now take $\{y_0, \dots, y_N\} = \{1/8, 4/8, 9/8, 13/8, 19/8, 23/8, 28/8\}$

y	g(y)	y	g(y)	y	g(y)	y	g(y)
0	+ 0,0000	8/8	+ 0,5858	16/8	+ 0,0000	24/8	- 0,5858
1/8	+ 1,2041	9/8	+ 0,8990	17/8	+ 0,9755	15/8	- 0,2501
2/8	+ 1,9313	10/8	+ 0,8341	18/8	+ 1,4829	26/8	- 0,2483
3/8	+ 1,9564	11/8	+ 0,2703	19/8	+ 1,3055	27/8	- 0,7038
4/8	+ 1,4142	12/8	- 0,5858	20/8	+ 0,5858	28/8	- 1,4142
5/8	+ 0,7038	13/8	- 1,3055	21/8	- 0,2703	29/8	- 1,9564
6/8	+ 0,2483	14/8	- 1,4829	22/8	- 0,8341	30/8	- 1,9313
7/8	+ 0,2501	15/8	- 0,9755	23/8	- 0,8990	31/8	- 1,2041
8/8	+ 0,5858	16/8	+ 0,0000	24/8	- 0,5858	32/8	+ 0,0000

Table I. Values of $g(x)$ in 4 decimals.

Lemma 3. Let f be a real valued function and put

$$A(x,h) = \frac{1}{h} (f(x) + f(x+h) - 2f(x+h/2))$$

If a positive number $M \geq 0$ exists so that for each $x_0 \in \mathbb{R}$ and $\delta > 0$ there exists a number h with $0 < h < \delta$, and a point $x \in \mathbb{R}$ so that $x \leq x_0 \leq x + h$ and

$$|A(x,h)| \geq M$$

then f has nowhere a finite derivative.

Proof: We have

$$\begin{aligned} A(x,h) &= \frac{f(x+h) - f(x)}{h} - \frac{f(x+h/2) - f(x)}{h/2} = \\ &= \frac{f(x+h) - f(x+h/2)}{h/2} - \frac{f(x+h) - f(x)}{h} . \end{aligned}$$

It follows that $A(x,h)$ is the difference between the increment-ratios of f on the intervals $[x, x+h]$ and $[x, x+h/2]$ resp. $[x+h/2, x+h]$ and $[x, x+h]$. For each point x_0 between x and $x+h$ there are two intervals $[a_1, b_1]$ and $[a_2, b_2]$ so that

$$x \leq a_1, a_2 \leq x_0 \leq b_1, b_2 \leq x+h, a_1 \neq b_1, a_2 \neq b_2$$

and

$$\left| \frac{f(b_1) - f(a_1)}{b_1 - a_1} - \frac{f(b_2) - f(a_2)}{b_2 - a_2} \right| = |A(x,h)| .$$

Now let $x_0 \in \mathbb{R}$ and $\delta > 0$. Let M, h and x be chosen as above.

Then we have

$$x_0 - \delta < x \leq x_0 \leq x + h < x_0 + \delta \quad \text{and}$$

$$|A(x,h)| > M .$$

It follows that there exists a_1, a_2, b_1 and b_2 so that

$$x_0 - \delta \leq a_1, a_2 \leq x_0 \leq b_1, b_2 \leq x_0 + \delta, a_1 \neq b_1, a_2 \neq b_2$$

and

$$\left| \frac{f(b_1) - f(a_1)}{b_1 - a_1} - \frac{f(b_2) - f(a_2)}{b_2 - a_2} \right| = |A(x, h)| > M$$

By Lemma 1 this is impossible if f has a finite derivative in x_0 . It follows that f has nowhere a finite derivative.

Theorem: The function $f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \sin(2^n \pi x)$ has nowhere a finite derivative.

Proof: Let $A(x, h) = \frac{1}{h} (f(x) + f(x+h) - 2f(x+h/2))$

Then $A(x, h) = \sum_{n=0}^{\infty} A_n(x, h)$ where

$$\begin{aligned} A_n(x, h) &= \frac{1}{2^n} \cdot \frac{1}{h} (\sin(2^n \pi x) + \sin(2^n \pi(x+h)) - 2\sin(2^n \pi(x+h/2))) \\ &= \frac{1}{2^n} \cdot \frac{1}{h} \cdot 2(\sin(2^n \pi(x+h/2))) \cdot [\cos(2^n \pi h/2) - 1] \end{aligned}$$

For $h = 2^{-k}$ this gives

$$A_n(x, 2^{-k}) = -2^{-(n-k-1)} \cdot \sin(2^n \pi(x+2^{-k-1})) \cdot (1 - \cos(2^{n-k-1} \pi))$$

It follows that $A_n(x, 2^{-k}) = 0$ for $k \geq n+2$.

For $n \leq k-2$ we have

$$\begin{aligned} |A_n(x, 2^{-k})| &\leq 2^{-(n-k-1)} |\sin(2^n \pi(x+2^{-k-1}))| |1 - \cos(2^{n-k-1} \pi)| \leq \\ &\leq 2^{-(n-k-1)} \cdot 1 \cdot \frac{1}{2} (2^{n-k-1} \pi)^2 = \\ &= \pi^2 \cdot 2^{n-k-2} \end{aligned}$$

Therefore

$$\begin{aligned} \left| \sum_{n=0}^{k-2} A_n(x, 2^{-k}) \right| &\leq \sum_{n=0}^{k-2} |A_n(x, 2^{-k})| \leq \sum_{n=0}^{k-2} \pi^2 \cdot 2^{n-k-2} = \\ &= \frac{\pi^2}{8} \sum_{n=1}^{k-1} 2^{-n} < \frac{\pi^2}{8} \end{aligned}$$

For the remaining terms we have:

$$\begin{aligned}
A_{k+1}(x, 2^{-k}) + A_k(x, 2^{-k}) + A_{k-1}(x, 2^{-k}) &= \\
&= -(2\sin(2^{k+1}\pi(x+2^{-k-1}))) + 2\sin(2^k\pi(x+2^{-k-1})) + \\
&\quad + 2(2-\sqrt{2})\sin(2^{k-1}\pi(x+2^{-k-1})) = \\
&= -2 g(2^k(x+2^{-k-1})) = -2 g(2^k x + 1/2)
\end{aligned}$$

$g(y)$ is defined as in Lemma 2.

Putting $R(x, k) = \sum_{n=0}^{k-2} A_n(x, 2^{-k})$ we conclude

$$\begin{aligned}
A(x, 2^{-k}) &= -2.g(2^k x + 1/2) + R(x, k) \\
\text{with } |R(x, k)| &< \frac{\pi^2}{8}.
\end{aligned}$$

Now let $x_0 \in \mathbb{R}$ and $\delta > 0$. Take $M = 0,1$. Let $k \in \mathbb{N}$ be chosen so that $2^{-k} < \delta$. By Lemma 2 there exists a point $y \in \mathbb{R}$ so that

$$y \leq 2^k x_0 + 1/2 \leq y + 1 \quad \text{and} \quad |g(y)| > \frac{\pi^2}{16} + 0,05$$

Putting $x = 2^{-k}(y-1/2)$ we have

$$x \leq x_0 \leq x + 2^{-k} \quad \text{and also}$$

$$\begin{aligned}
|A(x, 2^{-k})| &\geq |-2 g(y)| - |R(x, k)| > \\
&> 2(\pi^2/16 + 0,05) - \pi^2/8 = 0,1
\end{aligned}$$

We conclude that the condition in Lemma 2 is fulfilled, hence f has nowhere a finite derivative.

The above proof is restricted to finite derivatives. It is known that $f(x)$ has an infinite derivative at a dense point set in $\mathbb{R}$.

§2. The history of the non-differentiable continuous function

Before the first examples of non-differentiable continuous functions were given, differentiability was believed to be a consequence of continuity, although a number of isolated singularities were possible. There are authors known to have tried to prove rigidly the truth of this conviction. For example AMPÈRE [2] has given a proof for the differentiability of a continuous function using the (also erroneous) "fact" that for such a function the domain can be decomposed into a collection of intervals, on which the function is monotone.

WEIERSTRASS' function is the first example that has been published. P. du BOIS-REYMOND [8] describes in 1875 this function which Weierstrass has send to him in a letter. The following comment on this "horrible" example illustrates the mathematical feeling of the time (see [8]):

"Noch manches Rätsel scheint mir die Metaphysik der Weierstrassen Funktionen zu bergen, und ich kann mich des Gedankens nicht erwehren dass hier tieferes Eindringen schliesslich vor eine Grenze unseres Intellects führen wird, ähnlich der in der Mechanik durch die Begriffe Kraft und Materie Gezogenen. Diese Funktionen scheinen mir um es kurz zu sagen, räumliche Trennungen zu setzen nicht wie die Rationalzahlen in unbegrenzt Kleinen, sondern in unendlich Kleinen. Doch es ist hier nicht der Ort, auf so kontroverse Fragen näher einzugehen".

CELERIER's example was first published after his death in 1890 [15]. He may have discovered it already in 1830 but the exact date is unknown. BOLZANO describes in an unpublished paper in 1834 an example of a continuous function where in between any two points where the function has no derivative, there is another such point. The function he describes has however no finite derivative anywhere. A description of the original manuscript is given in [42]. For a proof of the non-differentiability see [69].

RIEMANN is known to have claimed that the function $f(x) = \sum_{n=1}^{\infty} \frac{\sin n^2 x}{n^2}$ is a non-differentiable function but no proof of his is known. (see the remark in [8]). This example is much more difficult then the others.

HARDY proves that f has no finite derivative for all irrational values of x [36]. His proof is based on earlier results of Littlewood and himself established "by reasoning of a highly transcendental character". To my knowledge up to now no elementary proof for Riemann's function exists.

In 1900 E.H. MOORE [58] proved that the coordinate-functions of the space-filling curves constructed by PEANO [65] and HILBERT [38] are also examples of nowhere differentiable functions. Research in this field has been carried out also by A.N. SING [76,80].

The presented examples of non-differentiable continuous functions can be divided in different types.

Type I: These are functions of the type

$$f(x) = \sum_{n=0}^{\infty} a_n \phi(b_n x)$$

where $\{a_n\}$ is a sequence so that $\sum_{n=0}^{\infty} |a_n| < \infty$, $\{b_n\}$ is a rapidly increasing sequence of integers and ϕ is a periodical function.

For ϕ is mostly chosen one of the following functions:

$\sin x$, $\cos x$, $|\sin x|$, $\Delta(x)$ (= the distance of x to the nearest integer), or $2\Delta(x/2 + 1/4) - 1/2$ (which is a "piecewise linear sine-function").

This class contains as a special case the functions

$$f(x) = \sum_{n=0}^{\infty} a^n \phi(b^n x) \text{ with } \text{integral } b > 1 \text{ and } ab \geq 1.$$

WEIERSTRASS' and CELERIER's example are of this type.

Type II: These functions are constructed geometrically as the limit function f of a uniform convergent sequence of piecewise linear functions f_k . Sometimes f_{k+1} is derived from f_k by applying a fixed transformation on the maximal linear subsegments of (the graph of) f_k . Such a linear subsegment of f_k is

transformed into the union of a finite number of subsegments of f_{k+1} having "large" differences in their increment-ratio's. In general the functions f_k are defined in such a way that $f_j(x) = f_k(x)$ for $j > k$ if x is one of the "corners" of f_k . Thus we have for all "corners" x of f_k equality $f(x) = f_k(x)$. BOLZANO's function is a typical example of this type. The theory of these constructions has been developed for example by K. KNOPP [47].

Type III: These functions are defined by considering the integers x_i in the expansion of x in some base $b \geq 2$: $x = \sum_{i=1}^{\infty} x_i b^i$, and by defining $f(x)$ in terms of the integers x_i .

A well-known example of a function of this type is the function $f(x) = \sum_{n=1}^{\infty} 2^{-n} \Delta(2^n x)$. This function is in fact a representative for each of the three types. Its definition as a type III function is the following:

$$\text{Let } x = n + \sum_{n=1}^{\infty} 2^{-n} x_n \quad n \in \mathbb{N}, x_n \in \mathbb{N}, x_n > x_{n-1}$$

$$\text{then } f(x) = \sum_{i=1}^{\infty} (x_i - 2(i-1)) \cdot 2^{-n_i}.$$

There are proofs for the existence of non-differentiable continuous functions by topological argument. These proofs are applications of the Baire category theorem on the space of all continuous function on a closed interval. This approach into functional analysis has been developed after 1920 (BANACH).

Another development has been the construction of even "worse" examples of functions. Weierstrass' function has nowhere an infinite derivative but it has a left or a right derivative $\pm \infty$ at a dense subset. It was an open problem up to 1925 whether there existed a continuous function having nowhere a left or a right derivative. The first example of a function of this class has been given by BESICOVITCH [5]. E.D. PEPPER [66] provided a simplified description of Besicovitch function while

A.P. MORSE [59] presented an analytical definition.

All examples of this class are much more complicated than the "simple" non-differentiable functions. It has been proved that this class of Besicovitch functions is much coarser than the class of non-differentiable functions. The Besicovitch functions form a first category set while the class of all non-differentiable functions is a residual set of a nowhere dense set, and therefore a second category set. See for example S. SAKS [72].

The behaviour of the four Dini-derivatives^{*} of a continuous function has equally become a field of research (DENJOY, GARG). Also the point-sets $f^{-1}(a) \subset \mathbb{R}$ have been studied for non-differentiable f .

The methods used to prove non-differentiability for some function f depend on the type of the example f .

If f is a first type function $f = \sum_{n=0}^{\infty} f_n = \sum_{n=0}^{\infty} a_n \phi(b_n x)$ one usually calculates the difference quotient $\frac{1}{h}(f(x+h) - f(x))$ for suitable values of h —mostly $\pm 1/b_k$ and $\pm 1/2b_k$; one then has an expansion

$$\frac{1}{h}(f(x+h) - f(x)) = \sum_{n=0}^{\infty} \frac{1}{h}(f_n(x+h) - f_n(x)).$$

For a suitably chosen $h = h_{1,k}$ and $h = h_{2,k}$ the terms with $n \geq k+1$ vanish in this expansion while the sum $\sum_{n=0}^{k-1} \frac{1}{h}(f_n(x+h) - f_n(x))$ is uniformly bounded by some constant M . Further we have for the k -th term, for some $\delta > 0$

$$\left| \frac{1}{h_{1,k}}(f_k(x+h_{1,k}) - f_k(x)) - \frac{1}{h_{2,k}}(f_k(x+h_{2,k}) - f_k(x)) \right| > M + \delta.$$

As $h_{i,k} \rightarrow 0$ it follows that $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ is undefined.

^{*} These are the functions: $D^+(x) = \limsup_{h>0} \frac{1}{h}(f(x+h) - f(x))$,
 $D^-(x) = \limsup_{h<0} \frac{1}{h}(f(x+h) - f(x))$, $D_+(x) = \liminf_{h>0} \frac{1}{h}(f(x+h) - f(x))$
and $D_-(x) = \liminf_{h<0} \frac{1}{h}(f(x+h) - f(x))$, which are always defined.

For second type functions the non-differentiability is generally proved by means of the criterion given in Lemma 1. In formulating this criterion some authors forget the crucial condition $x - \delta \leq a_1$, $a_2 \leq x \leq b_1$, $b_2 \leq x + \delta$ which is necessary to make the Lemma true (see for example [13]). The criterion together with a warning that this condition is crucial is present in the works of T.J. STIELTJES [87].

In the proof of the non-differentiability of a second type function one takes for a_1 , a_2 and b_1 , b_2 values of x where for some $k \in \mathbb{N}$ the function f_k has corners (and thus generally $f(a_i) = f_k(a_i)$ and $f(b_i) = f_k(b_i)$). By construction the difference of the increment ratios
$$\frac{f(b_i) - f(a_i)}{b_i - a_i}$$
 does not tend to zero and from there follows the non-differentiability.

If f is a third type function one generally takes $h = b^{-N}$. This way the expansions of x and $x + h$ become equal, with the exception of the N -th place. Exact calculation of the difference quotient $\frac{1}{h} (f(x+h) - f(x))$ becomes possible and the result is found to be diverging.

Functions of the type $f(x) = \sum_{n=1}^{\infty} \frac{1}{a^n} \Delta(a^n x)$ are the most generally given examples. As mentioned before they belong to each of the three types. For $a = 10$ this function is generally attributed to VAN DER WAERDEN [95]. The elegant proof he describes in this paper is in fact the solution of HEYTING and BUZEMAN which he received upon presenting this function as a problem in the journal WISKUNDIGE OPGAVEN MET DE OPLOSSINGEN in 1930 [37]. This proof however fails for $a < 4$. The example $f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \Delta(2^n x)$ is given already in [90] and [49].

Weierstrass gives in his original example $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n x)$ the condition $ab > 1 + \frac{3}{2} \pi$. Many authors have tried to exchange this artificial condition by the more natural one $ab \geq 1$. HARDY [36] succeeded in proving that the condition $ab \geq 1$, $a, b \in \mathbb{R}$ was sufficient to exclude a finite derivative in any point for both the functions $\sum_{n=0}^{\infty} a^n \sin(b^n x)$ and $\sum_{n=0}^{\infty} a^n \cos(b^n x)$. An infinite derivative becomes impossible if $ab > 1$.

This result is more general than any of the results which have been proved by elementary methods. All elementary proofs restrict themselves to integral b .

In the list of references the papers marked T are unknown to me, as they were not available in Amsterdam. As far as these papers are mentioned in "Mathematical Review", volume and page are indicated.

The papers marked F contain functional-analytical treatment while the papers treating more generalised derivatives are marked D. Papers marked L contain an interesting list of references or other background information. A Roman figure in between parentheses indicates the type of a given example.

1. S. ABIAN, 1960
Fund. Math. 49, 181-182
A nowhere differentiable and except on a denumerable set everywhere
continuous function (III)
2. AMPÈRE
Ecole polyt. 13, 148
..... T
3. S. BANACH, 1931
Studia Math. 3, 174-179
Über die Bairesche Kategorie gewisser Funktionenmengen F
4. F.A. BEHREND, 1949
Proc. London Math. Soc. (2), 50, 463-481
Some remarks on the construction of continuous non-differentiable
functions (I)
5. A.S. BESICOVITCH, 1925
Bull. Acad. Sci. USSR, 19, 527-540
Diskussion der stetigen Funktionen im Zusammenhang mit der
Frage über ihre Differenzierbarkeit T
6. L. BIEBERBACH
Differential und Integralrechnung I, 107
Leipzig 1917 (II)
7. R.P. BOAS
A primer of real functions, 61-62, 115, 126
Cars. Math. Monographs 13, John Whily and Sons, 1960
8. P. du BOIS-REYMOND, 1875
Crelle Journal für Mathematik, 79, 21-37
Versuch einer Klassifikation der willkürlichen Funktionen reeller
Argumente (I) L
9. T. BRODEN, 1897
Crelle Journal für Mathematik, 118, 1-60
Beiträge zur Theorie der stetigen Funktionen einer reellen
Veränderlichen (II)

10. T. BRODEN, 1905
 Arkiv för Math. Astr. und Phys., 2, Nr. 2
 Über die elementaire Konstruktion sogenannter Kurven ohne
 Tangente (II)
11. BROMWHICH
 Theory of infinite series, p.490 (I) T
12. A.M. BRUCKNER, J.L. LEONARD, 1966
 Amer. Math. Monthly, 73, April, part II, 24-56
 Derivatives. D L
13. BURRILL, KNUDSON, 1969
 Real Variables, p.253
 Holt Rinehart and Winston Inc. (1969)
14. K.A. BUSH, 1952
 Amer. Math. Monthly, 59, 222-225
 Continuous functions without derivatives (III)
15. CELERIER, 1890
 Bull. de Sci. Math. (2), 14, 142-160
 Note sur les principes fondamentaux de l'Analyse (I)
16. E. CESÀRO, 1906
 Archiv. für Math. und Phys. (3), 10, 57-63
 Fonctions continues sans dérivées (III)
17. H.T. CROFT, 1962
 Proc. Cambridge Phil. Soc., 58, 225-228
 On the sum of differentiable functions
18. DARBOUX
 Ann. de l'Ecole Norm. (2), 4, 57-112 and 8, 195-202
 T
19. A. DENJOY, 1915
 Journ. Math. Pures Appl. (7), 1, 105-240
 Mémoire sur les nombres dérivées des fonctions continues D

20. J. DIEUDONNÉ, 1933
Mathesis, 47, 277-279
Sur une fonction continue sans dérivée T
21. U. DINI, 1877
Annali di Matematica (2), 8, 121
Su Alcuni funzioni che in tutto un intervallo non hanno mai
derivata T
22. DINI - LÜROTH, 1892
Theorie der Funktionen einer veränderlichen reellen Größe
Leipzig 1892, 205-229 T
23. G. FABER, 1907
Jahresberichte der Deutschen Math. Ver., 16, 538
Einfaches Beispiel einer stetigen nirgends differenzierbaren
Funktion (I, II)
24. G. FABER, 1908
Math. Annalen, 66, 81-93
Über stetige Funktionen I (II)
25. G. FABER, 1910
Math. Annalen, 69, 372-443
Über stetige Funktionen II (II)
26. FALANGA, 1921
Giorn. di Mat., 59, 137
..... T
27. K.M. GARG, 1961
Ganita, 12, 9-14
An analogue of Denjoy's Theorem D
28. K.M. GARG, 1962
Rev. Math. Pures Appl., 7, 663-671
On nowhere monotone functions II (Derivates at sets of power $\underline{c}$
and at sets of positive measure) D

29. K.M. GARG, 1963
 Rev. Math. Pures Appl., 8, 83-90
 On nowhere monotone functions III (functions of first and second species) D
30. K.M. GARG, 1963
 Acta Math. Acad. Sc. Hungar., 14, 187-195
 Applications of Denjoy analogue III (Distribution of various typical level sets) D
31. K.M. GARG, 1968
 Can. Journ. of Math., 20, 135-142
 On assymmetrical derivates of non-differentiable functions D L
32. GELBAUM - OLMSTED, 1964
 Counterexamples in Analysis (II)
33. Z. GEÖCZE, 1904
 Az. Ungvari All. Realisk. Evi. Ertesitöje
 (II) T
34. H. HAHN, 1917
 Jahresb. der Deutschen Math. Ver., 26, 281-284
 Über stetige Funktionen ohne Ableitung (II)
35. H. HANKEL, 1870
 Untersuchungen über die unendlich oft oszillierenden und unstetigen Funktionen T
36. G.H. HARDY, 1916
 Trans. of the Amer. Math. Soc., 17, 301-325
 Weierstrass' non-differentiable function (I) L
37. A. HEYTING, H.H. BUZEMAN, 1930
 Wiskundige Opgaven met de oplossingen, 15, opg. 50
 (solution to a problem posed by B.L. van der Waerden) (I, II)
38. D. HILBERT, 1891
 Math. Ann., 38, 459
 Über die stetige Abbildung einer Linie auf ein Flächenstück

39. T.H. HILDEBRANT, 1933
 Amer. Math. Monthly, 40, 547-548
 A simple continuous function with a finite derivative at no point T
40. E.W. HOBSON, 1907 (1926)
 The Theory of functions of a real variable and the theory of Fourier series
 Vol. II, Dover 1957, p.401-412, (I)
41. V. JARNIK, 1933
 Fund. Math., 21, 48-58
 Über die Differenzierbarkeit stetiger Funktionen F
42. M. JASEK, 1920
 Sitz. Ber. der K. Böhm. Ges. der Wiss.
 Aus dem handschriftlichen Nachlass Bernard Bolzano's (II)
43. S. KANTOR, 1957
 Math. Lapok, 8
 Über die Funktion von Z. Geöcze
 (see Math. Rev. 20 no. 6495) T
44. S. KANTOR, 1962
 Publ. Math. Debr., 9, 111-121
 Über die Konstruktion überall stetiger nirgends-differenzierbaren Funktionen (II)
45. K. KIESWETTER, 1966
 Math. Phys. Semesterber., 13, 216-221
 Ein einfaches Beispiel für eine Funktion welche überall stetig und nirgends differenzierbar ist (II)
46. K. KNOPP, 1917
 Jahresber. der Deutschen Math. Ver., 26, 278-280
 Einfaches Beispiel einer stetigen nirgends differenzierbaren Funktion (II)
47. K. KNOPP, 1918
 Math. Zeitschrift, 2, 1-26
 Ein einfaches Verfahren zur Bildung stetiger nirgends differenzierbaren Funktionen (II) L

48. H. VAN KOCH, 1906
Acta Math., 30, 145-174
Une Méthode géométrique élémentaire pour l'étude de certaines questions de la théorie des courbes planes
49. G. LANDSBERG, 1908
Jahresber. der Deutschen Math. Ver., 17, 46-51
Über die Differenzierbarkeit stetiger Funktionen (I, II)
50. H. LEBESGUE, 1940
Enseignement Math., 38, 212-213
Une fonction continue sans dérivé (I)
51. LERCH, 1888
Crelle Journ. für Math., 103, 126-138
Über die Nicht-differenzierbarkeit gewisser Funktionen
52. F. MARCHIS, 1963
Ann. Univ. Timisoare Ser. Sci. Math. Fiz., No. 1, 187-190
On a continuous non-differentiable function (II) T
(see Math. Rev., 30, no. 4871)
53. S. MARCUS, 1962
Gaz. Math. Fiz. Ser., 14 (67), 79-80
An elementary example of a continuous function without a left or a right derivative at any point (III) T
(see Math. Rev., 27, no. 5874)
54. S. MAZURKIEWICZ - W. SIERPIŃSKI, 1924
Fund. Math., 6, 161-169
Sur un problème concernant les fonctions continues
55. S. MAZURKIEWICZ, 1931
Studia. Math., 3, 92-94
Sur les fonctions non dérivables F
56. M. MIKOLAS, 1956
Acta Sci. Math. Szeged., 17, 49-62
Construction des familles de fonctions partout continues non dérivables (I, II) L

57. S. MINAKSHI SUNDARAM, 1940
 Journ. Ind. Math. Soc., 4, 31-33
 On the roots of a continuous non-differentiable function T
 (see Math. Rev. 1, p.303)
58. E.H. MOORE, 1900
 Trans. Amer. Math. Soc., 1, 72-90
 On certain crinkly curves (II)
59. A.P. MORSE, 1938
 Trans. Amer. Math. Soc., 44, 496-507
 A continuous function with no unilateral derivatives
60. A.P. MORSE, 1954
 Proc. Amer. Math. Soc., 5, 126-130
 Dini derivatives of continuous functions D
61. B.N. MUKHOPADHYAY, 1934
 Bull. Calcutta Math. Soc., 25, 179-184
 On some generalisations of Weierstrass' non-differentiable function T
62. NAKANISHI SHIZU, 1964
 Proc. Japan. Acad., 40, 14-18
 Sur une fonction continue que est partout non-dérivable T
 (see Math. Rev. 29, no. 2338)
63. W. ORLICZ, 1947
 Fund. Math., 34, 45-60
 Sur les fonctions continues non-dérivables F
64. E. PASCAL, 1895
 Esercizi critici di calcolo differenziale e integrale
 Milano 1895, 85-128 T L
65. G. PEANO, 1890
 Math. Annalen, 36, 157
 Sur une courbe qui remplit toute une aire plane (II)

66. E.D. PEPPER, 1928
Fund. Math., 12, 244-253
On continuous functions without a derivative (II)
67. G. PIRANIAN, 1966
Amer. Math. Monthly, 73, April part II, 57-61
The set of non-differentiability of a continuous function D
68. M.B. PORTER, 1919
Bull. Amer. Math. Soc., 25, 176-180
Derivateless continuous functions (I)
69. K. REYCHLIK, 1921
Sitz.ber. der K. Böhm Ges. der Wiss.
Die Bolzanosche Funktion (II)
70. G. DE RHAM, 1967
Enseignement Math. (2), 3, 71-72
Sur un exemple de fonction continue sans dérivée (II)
71. G. DE RHAM, 1961
Celebrazioni Archimedee del sec XX (Siracosa), Vol II, 61-65
Ed. "odersi" Gubbio.
Sur quelques fonctions differentiable dont toutes les valeurs sont
des valeurs critiques. T
(see Math. Rev., 27, no. 1551)
72. S. SAKS, 1932
Fund. Math., 19, 211-219
On the functions of Besicovitch in the space of all continuous
functions. F
73. H. SCHERER, 1938
Semesterber zur Pflege des Zusammenhangs von Universität und
Schule (Munster), 12, 39-49
..... (III) T
74. SCHWARZ, 1873
Arch. des sciences de la bibliotheque universelle
..... T

75. W. SIERPIŃSKI, 1914
Bull. Acad. Sci. Cracov. (1914), 112
..... T
76. A.N. SINGH, 1924
Proc. Ben. Math. Soc., 6, 1-6
On space filling curves (II)
77. A.N. SINGH, 1925/26
Bull. Calcutta Math. Soc., 16, 101-108
Some new theorems on the derivatives of a function T
78. A.N. SINGH, 1930
Bull. Calcutta Math. Soc., 22, 1-8
On the derivates of a function T
79. A.N. SINGH, 1931
Bull. Calcutta Math. Soc., 23, 45-56
On the existence of a derivative T
80. A.N. SINGH, 1931
Proc. Benares Math. Soc., 13, 1-17
On the method of construction and properties of a class of
non-differentiable functions (I, II)
81. A.N. SINGH, 1935
The theory and the construction of non-differentiable functions T
Lucknow (1935)
82. A.N. SINGH, 1941
Proc. Benares Math. Soc. (N.S.), 3, 55-69
On functions without one-sided derivatives I (analytical
consideration of Besicovitch' function without one-sided
derivatives).
(see Math. Rev. 5, p.175) T
83. A.N. SINGH, 1942
Proc. Benares Math. Soc. (N.S.), 4, 95-108
On functions without one-sided derivatives II T
(see Math. Rev. 5, p.232)

84. A.N. SINGH, 1945
Fund. Math., 33, 106-107
On infinite derivatives (III)
85. STEINHAUS, 1929
Stdu. Math., 1, 51-81
Anwendungen der Funktionalanalyse auf einige Fragen der
reellen Funktionentheorie. F
86. STEINITZ, 1899
Math. Annalen, 52, 58-69
Stetigkeit und Differantialquotient (II)
87. T.J. STIELTJES, 1882
Nieuw Archief voor Wiskunde, 9, 106-111
Enige opmerkingen over de afgeleiden van een functie van één
variabele
(see also Complete Works I, p.67-72, Groningen 1914)
88. W.C. SWIFT, 1961
Amer. Math. Monthly, 68, 653-655
Simple constructions of non-differentiable and space filling
curves (III)
89. P. SZÁZ, 1960
Publ. Math. Debr., 7, 1-19
Einfache Herstellung einer Klasse von nirgends differenzierbaren
stetigen Funktionen auf Grund eines elementaren Satzes der
analytischen Geometrie
90. T. TAGAKI, 1904
Journ. of the phys. school in Tokyo, 14, 1-2
A simple example of continuous function without derived
function (III?) T
(see also Y. Mikami, Math. Papers of the Far East Leipzig 1910,
p.108)
91. R. TAMBS LYCHE, 1940
Enseignement Math., 38, 208-212
Une fonction continue sans dérivée (II, III)

92. E. TARNAWSKI, 1955
Fund. Math., 42, 11-37
Continuous functions in the logarithmic-power classifications
according to Hölders conditions (I)
93. E. TARNAWSKI, 1962
Enseignement Math. (2), 8, 271-274
Une simple démonstration de la non-dérivabilité de la fonction
du type de Weierstrass (I)
94. E.C. TICHMARSH, 1932 (1939)
The theory of functions (ed. II), Ch. XI, p.350-354
95. B.L. VAN DER WAERDEN, 1930
Math. Zeitschrift, 32, 474-475
Ein einfaches Beispiel einer nicht-differenzierbaren stetigen
Funktion (I, II, III)
96. WEIERSTRASS, 1890
Abhandlung aus der Funktionenlehre, p.92 (I)
(see also Complet Works II, p.223)
97. H. WHITNEY, 1935
Duke Math. Journal, 1, 514-517
..... T
98. WIENER, 1881
Crelle Journ. für Math., 90, 221-252
Geometrische und analytische Untersuchungen der Weierstrass'schen
Funktion (I)
99. W. WUNDERLICH, 1952
Elem. Math., 7, 73-79
Eine überall statige und nirgends differenzierbare Funktion (III)
100. G.C. YOUNG, 1916
Quart. Journ. Math. Oxford, 47, 127-175
On infinite derivatives T

101. W.H. YOUNG, 1908
Messenger of Mathematics, 38, 65-69
On the derivatives of non-differentiable functions T
102. Z. ZAKORSKI, 1941
Math. Sb. M.S., 2 (51), 487-510
Punktmengen, in welchen eine stetige Funktion nicht
differenzierbar ist. D
(Russian with German summary)
103. K. ZELLER, 1953
Math. Phys. Semesterber, 3
Kondensation von Singularitäten F
104. A. ZIGMUND, 1945
Duke Math. Journal, 12, 47-76
Smooth functions D